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Note

Irredundance number versus domination number

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Abstract

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The domination number $\gamma(G)$ and the irredundance number $\mathrm{ir}(G)$ of a graph G have been considered by many authors from a graph-theoretic or from an algorithmic point of view. In this graph-theoretic paper the infimum of all quotients $\mathrm{ir}(G)/\gamma(G)$ is investigated. It is well known that $\mathrm{ir}(G) \leq \gamma(G)$ holds for all undirected graphs G. We show that $\frac{2}{3}$ is the infimum of all quotients $\mathrm{ir}(T)/\gamma(T)$ in which T is a tree. Furthermore, there is no tree that attains the infimum. An analogous result for graphs is already known.

1. Domination and irredundance

$|A|$ denotes the cardinality of a set A. Let $G = (V, E)$ be a finite, undirected graph with neither loops nor multiple edges. V denotes the set of vertices, and E is the set of edges. For each $x \in V$ let $N(x)$ be the set of all vertices adjacent to x. Further, we define $N[x] = N(x) \cup \{x\}$. For $W \subseteq V$ we define $N[W] = \bigcup_{x \in W} N[x]$. $D \subseteq V$ is called a dominating set if $N[D] = V$. The cardinality $\gamma(G) = |D|$ of a minimum dominating set D is called the domination number of G.

Let be $I \subseteq V$. A vertex $x \in I$ is said to be redundant in I if $N[x] \subseteq N[I - \{x\}]$. Otherwise x is said to be irredundant in I. Finally, I is called an irredundant set if all $x \in I$ are irredundant in I. Otherwise I is a redundant set.

Obviously, each subset of an irredundant set is irredundant. A maximal irredundant set I of minimum cardinality is called an ir-set, and $\mathrm{ir}(G) = |I|$ is called the irredundance number of G.

Lemma [2, 6]. *Each minimum dominating set is maximal irredundant.*

Proof. Let $D \subseteq V$ be a minimum dominating set, and $x \in D$ redundant in D. Hence $N[x] \subseteq N[D - \{x\}]$. So $y \in N[D]$ implies $y \in N[D - \{x\}]$, and consequently, $D - \{x\}$ is a dominating set. This contradicts to the assumption. Now, let $D \cup \{z\}$ be irredundant for some $z \in V$. Then $N[z] \nsubseteq N[D]$, hence D cannot be dominating. □

Corollary. $\mathrm{ir}(G) \leq \gamma(G)$ *holds for all graphs* G.

The concept of irredundance in graphs was introduced in [3] while studying the domination number of graphs. A result similar to the present one was shown in [1] and [2]: $\inf \mathrm{ir}(G)/\gamma(G) = \frac{1}{2}$ where G is an arbitrary graph. The paper [5] is a survey about the first irredundance results. Many references in this field can also be found in [4]. The present paper was immediately stimulated by [6, proof of theorem 1].

2. The infimum of $\mathrm{ir}(T)/\gamma(T)$ for trees

Let $T_k = (V_k, E_k)$ be the tree defined as follows (Fig. 1):

$$V_k = \{x_{i1}, \ldots, x_{i6}: i = 1, \ldots, k\} \cup \{y_i: i = 1, \ldots, k-1\}.$$

$$\begin{aligned} E_k = \{&(x_{i1}, x_{i2}), (x_{i2}, x_{i3}), \ldots, (x_{i5}, x_{i6}): i = 1, \ldots, k\} \\ &\cup \{(x_{i3}, y_i), (y_i, x_{i+1,3}): i \text{ odd}\} \\ &\cup \{(x_{i4}, y_i), (y_i, x_{i+1,4}): i \text{ even}\}. \end{aligned}$$

One can easily verify that $I_k = \{x_{i3}, x_{i4}: i = 1, \ldots, k\}$ is maximal irredundant in T_k. So we have $\mathrm{ir}(T_k) \leq 2k$.

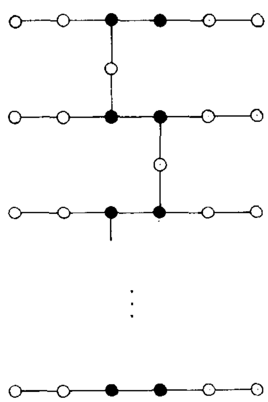

Fig. 1.

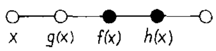

Fig. 2.

Now we want to find a minimum dominating set $D_k \subseteq V_k$. W.l.o.g. suppose that $x_{i2}, x_{i5} \in D_k$ ($i = 1, \ldots, k$), since these vertices are adjacent to the leaves of T_k. In addition we need $k-1$ vertices to dominate all the y_i. It results $\gamma(T_k) = 3k-1$, and $\mathrm{ir}(T_k)/\gamma(T_k) \leqslant 2k/(3k-1)$.

We obtain that for each real $\varepsilon > 0$ there exists a tree T with $\mathrm{ir}(T)/\gamma(T) < \frac{2}{3} + \varepsilon$. On the other hand, $\frac{2}{3}$ is a lower bound:

Theorem 1. *For all trees T we have* $\mathrm{ir}(T)/\gamma(T) > \frac{2}{3}$.

Proof. Let $T = (V, E)$ be a tree so that $\gamma(T) - \mathrm{ir}(T) = c > 0$, and $I \subseteq V$ an ir-set. We set $U = V - N[I]$. Since $U \cup I$ is dominating, we have $|U| \geqslant c$.

Consider an arbitrary $x \in U$, x cannot be redundant in $I \cup \{x\}$, thus there must exist a vertex $f(x) \in I$ being redundant in $I \cup \{x\}$. (If there are several such vertices, we choose an arbitrary of them to be $f(x)$.) Clearly, the distance between x and $f(x)$ must be 2. We denote by $g(x)$ the unique vertix adjacent to both x and $f(x)$. Further, $f(x)$ must be adjacent to some vertex $h(x) \in I$. (Fig. 2).

Assume $x_1, x_2 \in U$, $x_1 \neq x_2$, and $f(x_1) = f(x_2)$. We distinguish between the two cases (a) and (b) of Fig. 3.

In Case (a), $g(x_2)$ must be adjacent to a vertex of I distinct from $f(x_1)$, since $f(x_1)$ should be redundant in $I \cup \{x_1\}$. Note that $f(x_2)$ cannot be simultaneously irredundant in I and redundant in $I \cup \{x_2\}$. So Case (a) is impossible. We have shown that $f(x_1) = f(x_2)$ implies $g(x_1) = g(x_2)$. It follows

$$|\{g(x)\colon\ x \in U\}| \leqslant |\{f(x)\colon\ x \in U\}|.$$

We define

$$J = (I - \{f(x)\colon\ x \in U\}) \cup \{g(x)\colon\ x \in U\} \quad \text{and} \quad W = V - N[J].$$

It is evident that $|J| \leqslant |I|$ and $|W| \geqslant c$.

Consider $y \in W$. Since $U \subseteq N[J]$ we have $U \cap W = \emptyset$, hence $y \in N[I]$. The case

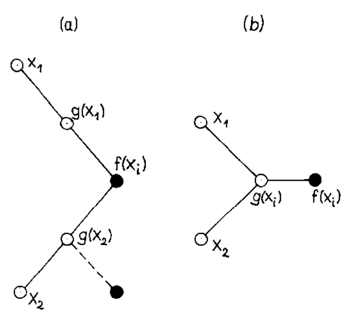

Fig. 3.

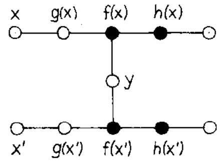

Fig. 4.

$y \in I$ is impossible. So y is adjacent to some vertex $f(x)$. Since $f(x)$ is redundant in $I \cup \{x\}$, y must be adjacent to another vertex $f(x') \in I$. (Fig. 4).

Now we construct a graph $T^* = (V^*, E^*)$ in the following way: Delete from T all vertices $y \in W$ together with their incident edges. The remaining graph has the connected components $C_1, \ldots, C_r$. We define

$$V^* = \{C_1, \ldots, C_r\} \quad \text{and} \quad E^* = W = \{y_1, \ldots, y_{|W|}\}.$$

Because of the situation shown in Fig. 4 there exist at least two components C_i, C_j so that y_1 is adjacent to some vertex from each of these components. (If there are more than two such components, we arbitrarily choose two of them.) We set $y_1 = (C_i, C_j) \in E^*$. Analogously we proceed with $y_2, \ldots, y_{|W|}$. Since T is a tree, the obtained graph T^* will be circuit-free. T^* has at least c edges. Hence the number of non-isolated vertices of T^* is at least $c + 1$. Further, each non-isolated C_i in T^* contains at least two vertices of I, since C_i contains some $f(x)$, and $f(x)$, $h(x)$ belong to the same component (Fig. 4). Now we can complete the proof: $\mathrm{ir}(T) = |I| \geq 2(c + 1)$, which implies

$$\mathrm{ir}(T) \geq 2\gamma(T) - 2\mathrm{ir}(T) + 2, \qquad \mathrm{ir}(T)/\gamma(T) \geq \tfrac{2}{3}(1 + 1/\gamma(T)),$$
$$\mathrm{ir}(T)/\gamma(T) > \tfrac{2}{3}. \qquad \square$$

We remark that the T_k defined above are caterpillars of hair length 2. So the infimum of $\mathrm{ir}(T)/\gamma(T)$ for trees can already be approximated by such caterpillars. On the contrary, for caterpillars of hairlength 1 it holds $\mathrm{ir}(T) = \gamma(T)$. This follows from [6] or from our proof.

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